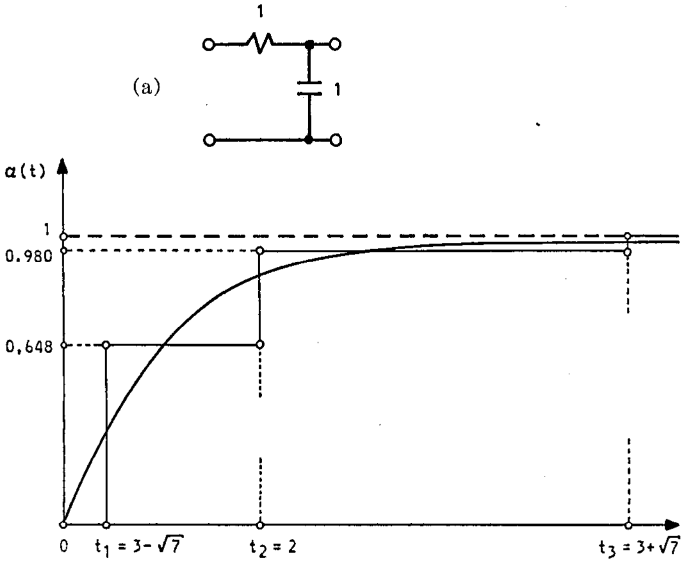

(b)

Fig. 4.

$$\mu_{i+1} = -\lambda_{i+1}\frac{N_{i+1}}{M_{i+1}} = -\lambda_{i+1}^2 N_{i+1}$$

$$(i = 1, 2, \cdots, n-1). \quad (22)$$

Finally we state the following important remark. At the arbitrarily chosen time instant $t = \xi$ the unit-step response satisfies the following inequalities provided by the theory of moments[6]

$$0 \leqq \alpha(\xi) - \alpha_n^*(\xi - 0) \leqq \frac{1}{Q'_{n+1}(\xi)Q_n(\xi) - Q'_n(\xi)Q_{n+1}(\xi)}.$$

Note that $Q_{n+1}(\xi)$ and $Q_n(\xi)$ may be computed through the algebraic recursive procedure indicated above from the $2n$ first moments of the impulse response. The number:

$$\chi_n(\xi) = \frac{1}{Q'_{n+1}(\xi)Q_n(\xi) - Q'_n(\xi)Q_{n+1}(\xi)}$$

gives the maximum error with which $\alpha(t)$ may be evaluated at $t = \xi$ from the first $2n$ moments.

Example

Consider the first four moments generated by the unit impulse voltage response of the RC network in Fig. 4(a).

$$m_0 = 1, \quad m_1 = 1, \quad m_2 = 2, \quad m_3 = 6, \quad m_4 = 24.$$

We will evaluate the function $\alpha_2^*(t)$ corresponding to these moments. Take $\xi = 2$. Following the recursive procedure, as presented previously, we easily get:

$$\lambda_2 = 1 \qquad \mu_2 = 3 \qquad \lambda_3 = \tfrac{1}{4}$$

$$Q_1(t) = t - 1, \quad Q_2(t) = t^2 - 4t + 2, \quad P_2(t) = t - 3$$

$$Q(t) = \left[\tfrac{1}{4}(t-2) + \frac{1}{-2}\right](t^2 - 4t + 2)$$

$$- (t - 1) = \tfrac{1}{4}(t-2)[t^2 - 6t + 2]$$

$$P(t) = [\tfrac{1}{4}(t-2) - \tfrac{1}{2}](t-3)$$

$$- 1 = \left(\frac{t}{4} - 1\right)(t - 3) - 1.$$

From the equation $Q(t) = 0$ we get:

$$t_1 = 3 - \sqrt{7} \qquad t_2 = 2(=\xi) \qquad t_3 = 3 + \sqrt{7}$$

and from relation (8)

$$A_1 = \frac{1}{3} - \frac{5\sqrt{7}}{42}, \qquad A_2 = \tfrac{1}{3}, \qquad A_3 = \frac{1}{3} + \frac{5\sqrt{7}}{42}.$$

The function $\alpha_2^*(t)$ and the precise unit-step response of the network

$$\alpha(t) = (1 - e^{-t})\mu(t)$$

are drawn in Fig. 4(b).

E. N. Protonotarios[2]
Bell Telephone Labs. Inc.
Holmdel, N. J.

O. Wing
Dept. of Elec. Engrg.
Columbia University
New York, N. Y.

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[2] Formerly with the Dept. of Elec. Engrg., Columbia University, New York, N. Y.

Circuit Transformations for Crystal Ladder Filters

The standard realization of parametric bandpass filters, crystal filters in particular, does permit a substitution of piezo-electric resonators only if consecutive attenuation poles above and below the passband alternate.[1] However, it is often more practical to remove first all of the attenuation poles on one side and then those of the other side. Typical configurations are shown in another work[2] for the extreme conditions that attenuation poles are provided only above or below the passband. Network transformations must then be applied in order to achieve a structure in which piezo-electric resonators could be substituted for the actual resonators, potentially.

Figures 1 and 3 demonstrate methods which may be used to perform such transformation and, simultaneously, yield equal inductances as an additional convenience of practical importance. Both make use of the particular Norton transformations shown in Fig. 2. Configuration (A) showing a Norton transformation applied to a capacitor in the series branch, is employed between Step 2 and 3 of Fig. 1. Configuration (B) of Fig. 2 showing a Norton transformation applied to a shunt capacitor, is employed between Step 2 and 3 of Fig. 3.

In both figures, the transformation ratios are arbitrary. However, in order to satisfy the additional restraint of equal inductances of all inductors, these ratios are well determined. For the configurations

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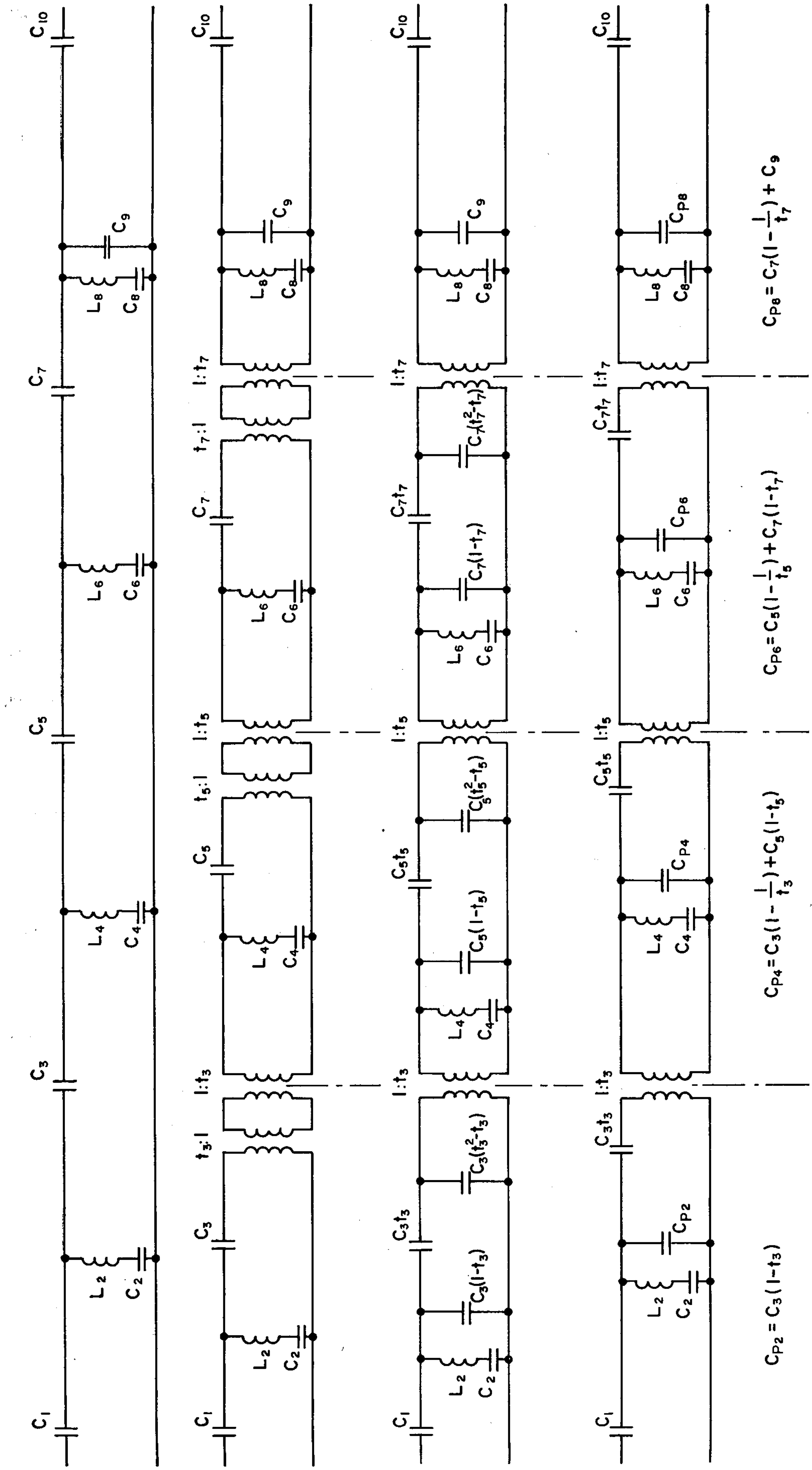
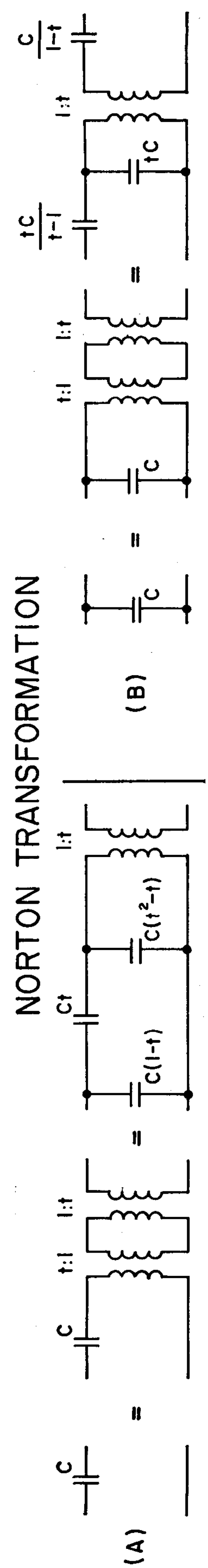


$C_{P2} = C_3(1-t_3)$
$C_{P4} = C_3(1-\frac{1}{t_3}) + C_5(1-t_5)$
$C_{P6} = C_5(1-\frac{1}{t_5}) + C_7(1-t_7)$
$C_{P8} = C_7(1-\frac{1}{t_7}) + C_9$

Fig. 1.



NORTON TRANSFORMATION
(A)
(B)

Fig. 2.

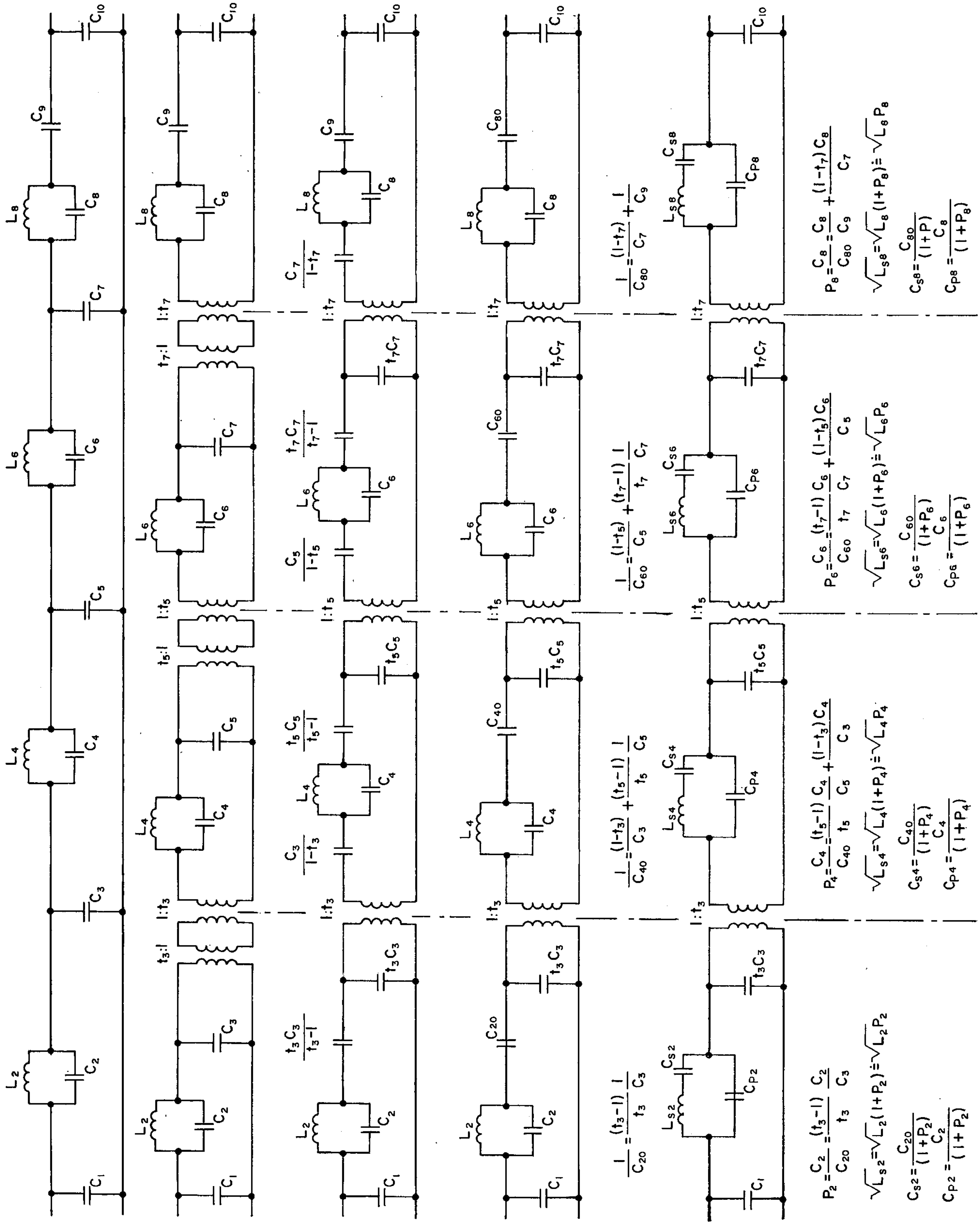


Fig. 3.

of Fig. 1, these conditions are:

$$t_3 = \sqrt{\frac{L_4}{L_2}}; \quad t_5 = \sqrt{\frac{L_6}{L_4}}; \quad t_7 = \sqrt{\frac{L_8}{L_6}}. \tag{1}$$

For the configuration of Fig. 3, the conditions are more complicated, due to the fact that two types of transformations are needed to derive the final configuration from the initial circuit. Equal series-inductances in the final configuration postulate the following conditions:

$$t_3\sqrt{L_{s2}} = \sqrt{L_{s4}}; \quad t_5\sqrt{L_{s4}} = \sqrt{L_{s6}}; \quad t_7\sqrt{L_{s6}} = \sqrt{L_{s8}} \tag{2}$$

thus approximately (for sufficiently large Pi's):

$$t_3\rho_2\sqrt{L_2} = \rho_4\sqrt{L_4}; \quad t_5\rho_4\sqrt{L_4} = \rho_6\sqrt{L_6}; \quad t_7\rho_6\sqrt{L_6} = \rho_8\sqrt{L_8}. \tag{3}$$

Substituting the pertinent expressions of Fig. 3 yields then the following three equations:

$$\sqrt{L_2}\,(t_3 - 1)\frac{C_2}{C_3} = \sqrt{L_4}\left[\left(\frac{t_5 - 1}{t_5}\right)\frac{C_4}{C_5} + (1 - t_3)\frac{C_4}{C_3}\right] \tag{4a}$$

$$\sqrt{L_4}\left[(t_5 - 1)\frac{C_4}{C_5} + (1 - t_3)t_5\frac{C_4}{C_3}\right] = \sqrt{L_6}\left[\left(\frac{t_7 - 1}{t_7}\right)\frac{C_6}{C_7} + (1 - t_5)\frac{C_6}{C_5}\right] \tag{4b}$$

$$\sqrt{L_6}\left[(t_7 - 1)\frac{C_6}{C_7} + (1 - t_5)t_7\frac{C_6}{C_5}\right] = \sqrt{L_8}\left[\frac{C_8}{C_9} + (1 - t_7)\frac{C_8}{C_7}\right]. \tag{4c}$$

These equations yield subsequently:

$$a_{31}(t_3 - 1)t_5 = (t_5 - 1)b_3 \tag{5a}$$

$$a_{51}(t_5 - 1)t_7 + a_{52}(1 - t_3)t_5t_7 = (t_7 - 1)b_5 \tag{5b}$$

$$a_{71}(t_7 - 1) + a_{72}(1 - t_5)t_7 = b_7 \tag{5c}$$

with

$$a_{31} = \left(\frac{C_2}{C_3}\sqrt{L_2} + \frac{C_4}{C_3}\sqrt{L_4}\right) \qquad b_3 = \frac{C_4}{C_5}\sqrt{L_4} \tag{6a}$$

$$a_{51} = \left(\frac{C_4}{C_5}\sqrt{L_4} + \frac{C_6}{C_5}\sqrt{L_6}\right); \qquad a_{52} = \frac{C_4}{C_3}\sqrt{L_4};$$

$$b_5 = \frac{C_6}{C_7}\sqrt{L_6} \tag{6b}$$

$$a_{71} = \left(\frac{C_6}{C_7}L_6 + \frac{C_8}{C_7}\sqrt{L_8}\right); \qquad a_{72} = \frac{C_6}{C_5}\sqrt{L_6};$$

$$b_7 = \frac{C_8}{C_9}\sqrt{L_8}. \tag{6c}$$

These coefficients depend only on the circuit elements of the initial circuit. The equations are then modified in the following manner:

$$(t_3 - 1)t_5 = (t_5 - 1)d_3 \quad \text{with} \quad d_3 = \frac{b_3}{a_{31}} \tag{7a}$$

$$(t_5 - 1)t_7 = (t_7 - 1)d_5 \quad \text{with} \quad d_5 = \frac{b_5}{a_{51} - d_3a_{52}} \tag{7b}$$

$$(t_7 - 1) = d_7 \quad \text{with} \quad d_7 = \frac{b_7}{a_{71} - d_5a_{72}}. \tag{7c}$$

Solving these equations starting with the last and proceeding toward the first yields the following solutions:

$$t_7 = (1 + d_7); \qquad t_5 = \left(1 + \frac{d_5 \cdot d_7}{1 + d_7}\right);$$

$$t_3 = \left(1 + \frac{d_3 \cdot d_5 \cdot d_7}{1 + d_7 + d_5 \cdot d_7}\right). \tag{8}$$

Both methods, Fig. 1 as well as Fig. 3, may be expanded to an arbitrary number of sections. The structure of the formulas for the t's suggests that a general formula for an arbitrary number of sections should be easy to find.

In both final structures, all inductances will become equal after an eventual removal of all transformers. It may occasionally happen, however, that some of the parallel capacities may become negative. This may be remedied by modifying some or all of the transfer ratios between sections. An alternative would be to alter the sequence of the attenuation poles.

Numerical examples for either transformation may be found in another work.[2]

E. Christian
ITT Telecommunications and
North Carolina State University
Raleigh, N. C.

Transient Response of Equal-Element Bandstop Filters

Introduction

The impulse and step responses of equal-element bandpass filters have been previously derived.[1] This correspondence extends those calculations to include the equal-element bandstop case.

Transient responses have been obtained with the aid of a computer and plotted for one through eight resonant circuits for many degrees of dissipation. All parameters have been normalized so that the curves are universal and can be used for equal-element filters of any bandwidth, center frequency, or unloaded Q. This information eliminates the use of steady-state amplitude response information for estimating transient response.

The transient response is derived from the voltage gain of the circuit; the input voltage takes into account generator loading. This is the most general case and is especially helpful where transistor sources are used. The analysis is done solely for equal generator and load resistances, which is the desired loading for an equal-element filter.[2]

Theoretical Considerations

Analysis is based on the transformation of the equivalent circuit of a bandstop filter to its lowpass prototype. Figure 1(a) shows the conventional lowpass prototype circuit and Fig. 1(b) shows it modified to include dissipation. These prototypes can be analyzed to predict the behavior of bandstop filters by using the transformation

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